Exam Answer Key.

1. a) $f'(x) = 3^x \ln 3 + 3\,\frac{1}{2}\,x^{-3/2} + \frac{1}{x^2 \ln 3}(2x) + \frac{1}{x^2}$

3

b) $f'(x) = (\sin^{-1}(x^2) + e^x)[3x^2 + \frac{1}{x}]$

3

$+ \left[\frac{1}{\sqrt{1-(x^2)^2}}(2x) + e^x\right](x^3 + \ln x)$

c) $f'(x) = 4\sec^3(\sin(e^x))\ \sec(\sin(e^x))\tan(\sin(e^x))$

3

$\cdot \cos(e^x) \cdot e^x$

d) $f'(x) = \dfrac{(x^3+e^{x^2})\left(\frac{1}{(x^2+1)^2+1}\right)(2x) - \tan^{-1}(x^2+1)(3x^2+e^{x^2}2x)}{(x^3+e^{x^2})^2}$

3

e) $y = (\sin 2x)^x \qquad \ln y = x\ln(\sin 2x)$

3

$\frac{1}{y}\frac{dy}{dx} = x\,\frac{1}{\sin(2x)}\cos(2x)(2) + \ln(\sin 2x)$

$\therefore \quad \frac{dy}{dx} = (\sin 2x)^x\left[\frac{x\,2\cos(2x)}{\sin 2x} + \ln(\sin 2x)\right]$

f) $f'(x) = [2(\cos x)^3 + \cos x - 1](-\sin x)$

2

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2. $y \ln x - y e^x + x = 1$ tangent @ $(1,0)$

$$y\left(\frac{1}{x}\right) + \ln x \frac{dy}{dx} - y e^x - e^x \frac{dy}{dx} + 1 = 0$$

(4) $$\frac{dy}{dx} = \frac{y e^x - 1 - y/x}{\ln x - e^x}$$

@ $(1,0)$ $$y' = \frac{0 - 1 - 0}{\ln(1) - e^1} = \frac{-1}{-e} = 1/e$$

$\therefore$ tangent line: $y - 0 = \frac{1}{e}(x-1)$

$\therefore$ $y = \frac{1}{e}(x-1)$.

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(4) a) $\int \left(3x^2 + e^{3x} + 3^x - \frac{3}{x} + \pi\right) dx$

$= x^3 + \frac{e^{3x}}{3} + \frac{3^x}{\ln 3} - 3\ln|x| + \pi x + C$

(4) b) $\int_0^1 \frac{x}{(x^2+5)^{1/3}} dx$ let $u = x^2+5$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$= \frac{1}{2}\int_5^6 u^{-1/3}\, du = \frac{1}{2} \frac{u^{2/3}}{2/3} \Big|_5^6$

$= \frac{3}{4}\left[6^{2/3} - 5^{2/3}\right]$.

(4) c) $\int \cos^4 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 \cancel{4}x\right) dx$

$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{\cancel{4}2}\right) dx$

$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$

$= \frac{1}{4}\left[\frac{3}{2}x + 2\frac{\sin 2x}{2} + \frac{1}{2}\frac{\sin 4x}{4}\right] + C.$

(4) d) $\int x \cot^{-1} x \, dx$

$u = \cot^{-1} x \qquad dv = x\,dx$

$du = \frac{-1}{x^2+1}\,dx \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\cot^{-1}x + \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$$

$$\begin{array}{r} 1 \\ x^2+1 \overline{\smash{)}\, x^2 \quad} \\ \underline{x^2+1} \\ -1 \end{array}$$

$$= \frac{x^2}{2}\cot^{-1}x + \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx$$

$$= \frac{x^2}{2}\cot^{-1}x + \frac{1}{2}\left[x - \tan^{-1}x\right] + C$$

(4) e) $\displaystyle\int \frac{5x^2-3x+7}{(x-1)(x^2+2)}\,dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+2}\,dx$

$\therefore\ 5x^2-3x+7 = Ax^2+2A + (Bx+C)(x-1)$

if $x=1$: $5-3+7 = 3A$ $\qquad \underline{A=3}$

x^2 : $5 = A+B$ $\qquad \therefore B=2$

$x=0$: $7 = 2A - C$ $\qquad C=-1$

$$\int \left(\frac{3}{x-1} + \frac{2x}{x^2+2} - \underbrace{\frac{1}{x^2+2}}\right) dx$$

if $x = \sqrt{2}\tan\theta$

$dx = \sqrt{2}\sec^2\theta\, d\theta$

$\displaystyle\int \frac{1}{x^2+2}\,dx = \int \frac{\sqrt{2}\sec^2\theta}{2\sec^2\theta}\,d\theta$

$\displaystyle = \frac{1}{\sqrt{2}}\theta = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$$= 3\ln|x-1| + \ln(x^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}(x/\sqrt{2}) + C$$

(4) f.) $\displaystyle\int \frac{x}{\sqrt{8x-x^2}}\,dx$

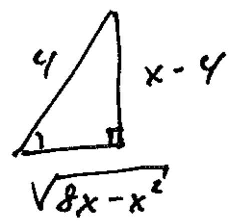
$$-x^2+8x = -(x^2-8x)$$
$$= -[(x-4)^2-16]$$
$$= 16-(x-4)^2$$

$$= \int \frac{x}{\sqrt{16-(x-4)^2}}\,dx$$

let $x-4 = 4\sin\theta$
$dx = 4\cos\theta\,d\theta$

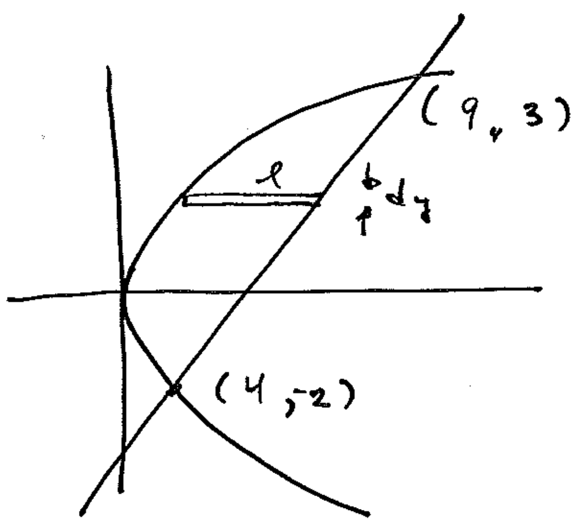
$$= \int \frac{(4\sin\theta+4)\,4\cos\theta}{4\cos\theta}\,d\theta = \int (4\sin\theta+4)\,d\theta$$

$$= -4\cos\theta + 4\theta + C$$



$$= -4\,\frac{\sqrt{8x-x^2}}{4} + 4\sin^{-1}\left(\frac{x-4}{4}\right) + C$$

(3) 4.



$$A = \int_{-2}^{3} l\,dy$$

$$= \int_{-2}^{3} (y+6-y^2)\,dy$$

$$= \frac{y^2}{2} + 6y - \frac{y^3}{3}\Bigg|_{-2}^{3}$$

$$= \frac{9}{2} + 18 - 9 - \left[2 - 12 + \frac{8}{3}\right] = 9 + \frac{9}{2} + 10 - \frac{8}{3}$$

$$= \boxed{\frac{125}{6}}$$

5.

(2) a)

$$\lim_{x \to -1^-} \frac{3x(x+3)}{(x+1)^2} = -\infty$$

$$\lim_{x \to -1^+} \frac{3x(x+3)}{(x+1)^2} = -\infty$$

$$\lim_{x \to \infty} \frac{3x^2+9x}{(x+1)^2} = 3$$

$$\lim_{x \to -\infty} \frac{3x^2+9x}{(x+1)^2} = 3$$

(3) b)

$$f'(x) = \frac{-3(x-3)}{(x+1)^3}$$

		-1		3	
-3			$-$		
$x-3$			$-$	|	$+$
$(x+1)^3$	$-$	⋮	$+$		
f'	$-$	⋮	$+$	|	$-$

$f(x)$ is

decr : $(-\infty, -1) \cup (3, \infty)$

incr : $(-1, 3)$

max @ $(3, 27/8)$

$$f''(x) = \frac{6(x-5)}{(x+1)^4}$$

		-1		5	
$6(x-5)$			$-$	|	$+$
$(x+1)^4$	$+$	⋮	$+$		
f''	$-$	⋮	$-$	|	$+$

$f(x)$ is

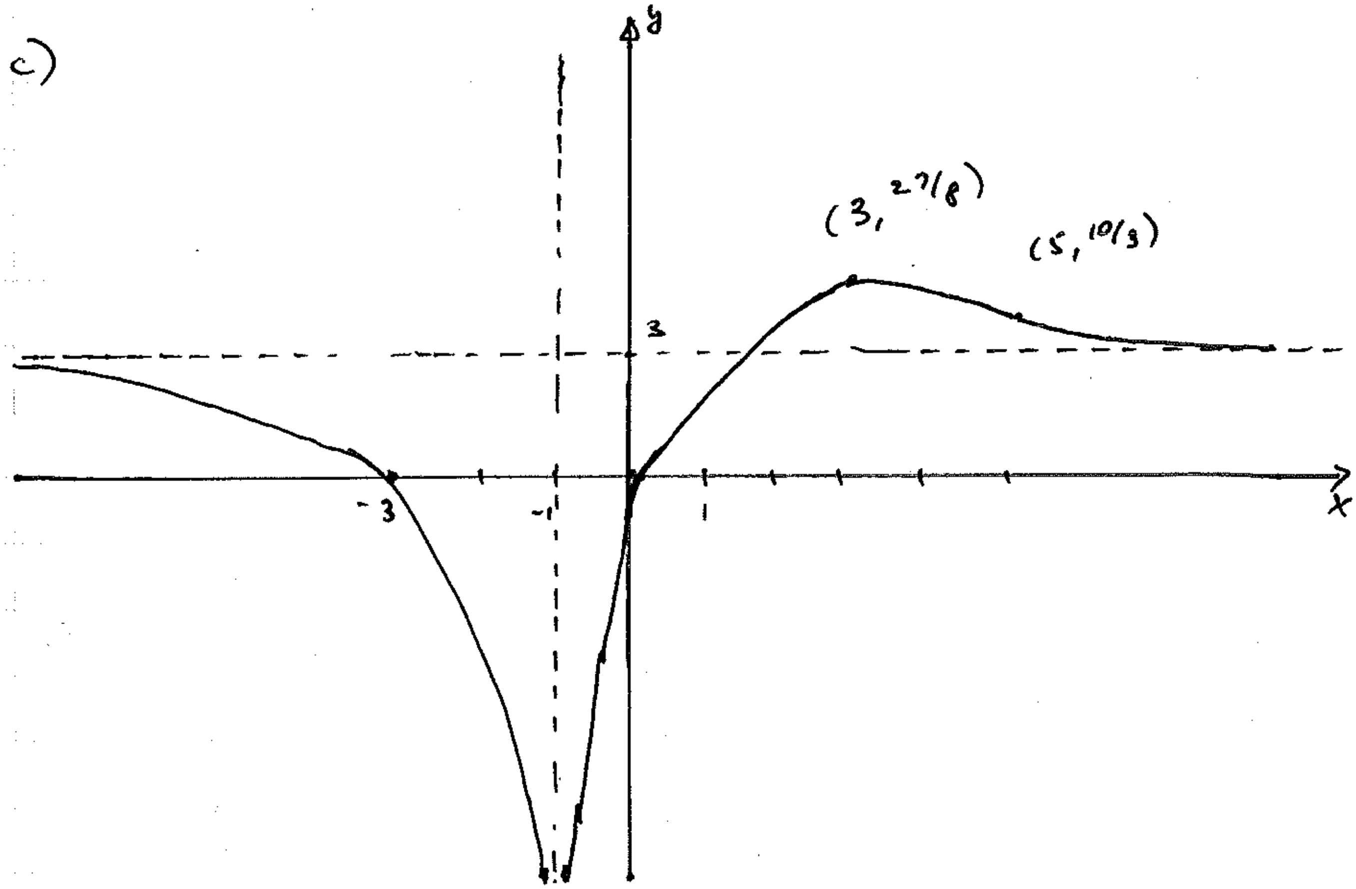
concave down : $(-\infty, -1) \cup (1, 5)$

concave up : $(5, \infty)$

Pt of inflection @ $(5, 10/3)$

(3)

c)



6.

(1) a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

(3) b) $f'(x) = \lim_{h \to 0} \frac{\sqrt{2 - 3(x+h)} - \sqrt{2-3x}}{h}$

$$= \lim_{h \to 0} \left[\frac{\sqrt{2-3x-3h} - \sqrt{2-3x}}{h} \right] \left[\frac{\sqrt{2-3x-3h} + \sqrt{2-3x}}{\sqrt{2-3x-3h} + \sqrt{2-3x}} \right]$$

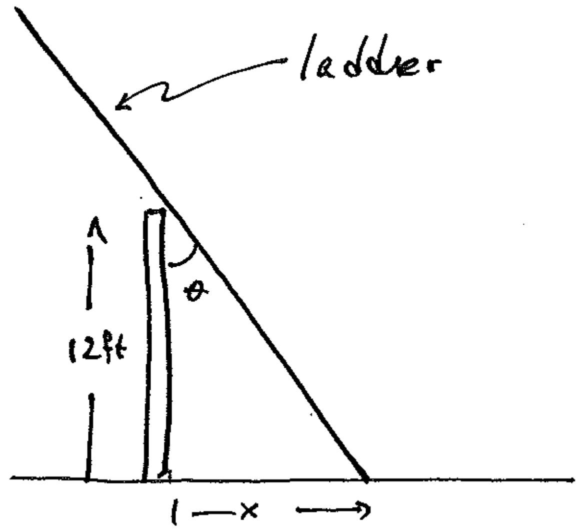
$$= \lim_{h \to 0} \frac{2 - 3x - 3h - 2 + 3x}{h \left[\sqrt{2-3x-3h} + \sqrt{2-3x} \right]}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h} \left(\sqrt{2-3x-3h} + \sqrt{2-3x} \right)}$$

$$= \frac{-3}{2\sqrt{2-3x}} .$$

7.

(5)



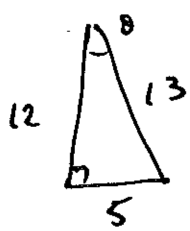
$$\left.\frac{d\theta}{dt}\right|_{x=5} = ?$$

$$\frac{dx}{dt} = 3 \text{ ft/sec}$$

$$\tan\theta = \frac{x}{12} \qquad \sec^2\theta \, \frac{d\theta}{dt} = \frac{1}{12}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{12}\frac{dx}{dt}\cos^2\theta$$

when $x = 5$

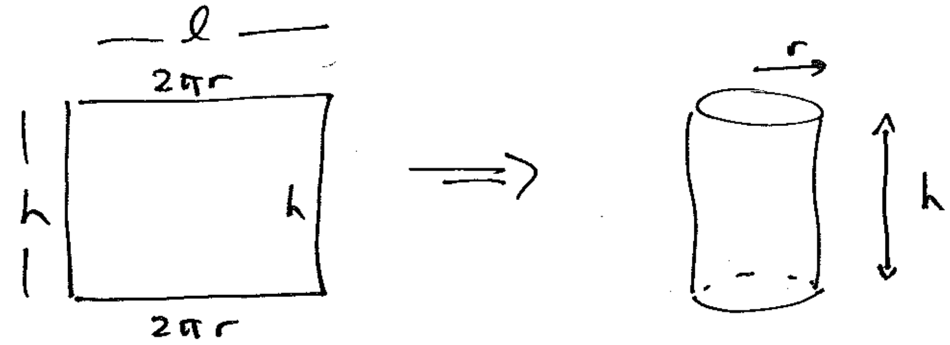


$$\cos\theta = \frac{12}{13}$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{1}{12}(3)\left(\frac{12}{13}\right)^2 = \frac{36}{169} \text{ rad/sec.}$$

8.

(15)



$$V = \pi r^2 h$$

$$2h + 4\pi r = 30 \text{ cm}$$
$$h = 15 - 2\pi r$$

$$V = \pi r^2 (15 - 2\pi r)$$

$$V = 15\pi r^2 - 2\pi^2 r^3 \qquad 0 \leq r \leq 15/2\pi$$

$$\frac{dV}{dr} = 30\pi r - 6\pi^2 r^2$$
$$= 6\pi r (5 - \pi r)$$

critical #'s:
$r = 0, \ 5/\pi$

$$V(0) = 0$$
$$V(15/2\pi) = 0$$
$$V(5/\pi) = \pi \left(\frac{5}{\pi}\right)^2 (15 - 10) = \frac{125}{\pi} \text{ cm}^3$$

$\therefore$ Rectangle has $r = 5/\pi$
$h = 5$
$l = 2\pi r = 10.$

9.

(2½) a) $\lim\limits_{x \to 1} \dfrac{-x^3+x^2-4x+4}{x^2-2x+1}$

$= \lim\limits_{x \to 1} \dfrac{-(x-1)(x^2+4)}{(x-1)(x-1)} = \lim\limits_{x \to 1} \dfrac{-(x^2+4)}{(x-1)} = \underline{\underline{d.n.e.}}$

since $\lim\limits_{x \to 1^-} \dfrac{-(x^2+4)}{(x-1)} \neq \lim\limits_{x \to 1^+} \dfrac{-(x^2+4)}{(x-1)}$

(2) b) $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x \sin 3x} = \lim\limits_{x \to 0} \dfrac{\left(\dfrac{\sin x}{x}\right)_{\to 1} \left(\dfrac{\sin x}{x}\right)_{\to 1}}{3 \left(\dfrac{\sin 3x}{3x}\right)_{\to 1}}$

$= \dfrac{1}{3}$

(2½) c) $\lim\limits_{x \to -\infty} \dfrac{(\sqrt{9x^6 - x})/x^3}{(x^3+1)/x^3}$

$= \lim\limits_{x \to -\infty} \dfrac{\sqrt{9x^6-x}/(-\sqrt{x^6})}{1 + \frac{1}{x^3}}$

$= \lim\limits_{x \to -\infty} \dfrac{-\sqrt{9 - 1/x^5}}{1 + 1/x^3} = \underline{-3}$

7

10.

(2) a) $\lim_{x \to 0} \frac{x^2}{1 - \cos 2x} = \lim_{x \to 0} \frac{2x}{2 \sin 2x} = \lim_{x \to 0} \frac{1}{2 \frac{\sin 2x}{dx}} = \frac{1}{2}$

(3) b) $y = \lim_{x \to \infty} (e^x + x)^{1/x}$

$\ln y = \lim_{x \to \infty} \frac{1}{x} \ln(e^x + x)$

$= \lim_{x \to \infty} \frac{\frac{1}{e^x + x} \cdot (e^x + 1)}{1} = \lim_{x \to \infty} \frac{e^x + 1}{e^x + x}$

$= \lim_{x \to \infty} \frac{e^x}{e^x + 1} = 1$

$\therefore \ln y = 1 \qquad \therefore \lim_{x \to \infty} (e^x + x)^{1/x} = e^1 = \underline{\underline{e}}.$

(3) 11. $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx = \lim_{b \to \infty} \int_1^b \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx$

$= \lim_{b \to \infty} \left[-2 e^{-\sqrt{x}} \Big|_1^b \right]$

$= \lim_{b \to \infty} \left(\underbrace{-2 e^{-\sqrt{b}}}_{\to 0} + 2 e^{-1} \right) = \boxed{+\frac{2}{e}}.$

12 a) If $f(x)$ is continuous on $[a,b]$

(2) and $f'(x)$ exists on (a,b)

then there is a pt $c \in (a,b)$

such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

b) Since $f'(x)$ exists in $[a,b]$

(3) then $f(x)$ is continuous in $[a,b]$.

$\therefore$ by the MVT, there is a pt $c \in (a,b)$

such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

but $|f'(c)| > 1$

$$\therefore \left| \frac{f(b)-f(a)}{(b-a)} \right| > 1$$

$$\therefore \frac{|f(b)-f(a)|}{|b-a|} > 1$$

$$\therefore |f(b)-f(a)| > |b-a| .$$

13. $f(x) = \begin{cases} \dfrac{x^2-1}{(x+3)(x-1)} & x < 1 \\ \dfrac{2-x}{2x} & x \geq 1 \end{cases}$

(1) a) domain : $x \in \mathbb{R}, \ x \neq -3$

b) 1) $f(x)$ does not exist at $x = -3$

(3) 2) @ $x = 1$:

$$\lim_{x \to 1^-} \frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}} = \frac{1}{2}$$

2

$$\lim_{x \to 1^+} \frac{2-x}{2x} = \frac{1}{2}$$

$$f(1) = \left.\frac{2-x}{2x}\right|_{x=1} = \frac{1}{2}$$

$\therefore$ $f(x)$ is continuous @ $x = 1$.

$\therefore$ intervals of continuity are
$(-\infty, -3), (3, \infty)$

14.

(1) a) for $f(x)$ to be continuous @ $x=a$ we have $\lim_{h \to 0} f(a+h) = f(a)$.

(2) b) <u>Proof:</u>

let 1) $f(a+h) = f(a+h)$

2) $f(a+h) = f(a+h) - f(a) + f(a)$

3) $f(a+h) = \dfrac{(f(a+h) - f(a))h}{h} + f(a)$

$(h \neq 0)$

then

$$\lim_{h \to 0} f(a+h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot h + f(a)$$

since $f'(a)$ exists (given)

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

$\therefore$

$$\lim_{h \to 0} f(a+h) = f'(a) \cdot 0 + f(a)$$

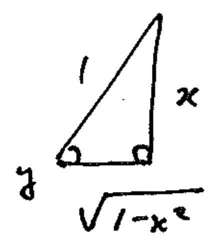
$$= f(a).$$

15. For $y = \sin^{-1}(x)$

(1) a) we get $\sin y = x \qquad -\frac{\pi}{2} \leqslant y \leqslant \pi/2$

(2) b) $\therefore \quad \cos y \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$



$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} .$$