MATH-32.1101/6 **Intro to Calculus – Final Examination**

April 9, 2008 - 9:00 a.m. **Duration:** 3 hours

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STUDENT NAME: ________________ STUDENT NO.: ________________

Special Instructions:

WRITE YOUR NAME IN THE SPACE ABOVE AND CIRCLE YOUR INSTRUCTOR'S NAME. There are sixteen questions. Answer all of the questions on these pages in the spaces following each question. Use the backs of pages for rough work if necessary. Calculators or similar electronic aids are **NOT permitted.**

The following table is for the marker's use.

1	2	3	4	5	6	7	8	
9	10	11	12	13	14	15	16	Total

Value

16 1. Differentiate the following: **[Do not simplify]**

(a) $f(x) = 8x^7 - 7x^{-8} + 8^x + \dfrac{2}{\sqrt[3]{x}} - \log_5 x + \ln 2e$

$f(x) = 8x^7 - 7x^{-8} + 8^x + 2x^{-\frac{1}{3}} - \log_5 x + \ln 2e$

$f'(x) = 56x^6 + 56x^{-9} + 8^x \ln 8 - \frac{2}{3}x^{-\frac{4}{3}} - \frac{1}{x\ln 5}$

(b) $f(x) = (\sec x - \csc x)(\tan x + \cot x)$

$f'(x) = [\sec x \tan x + \csc x \cot x](\tan x + \cot x) + (\sec x - \csc x)[\sec^2 x - \csc^2 x]$

Value

Differentiate the following: [Do not simplify]

(c) $f(x) = \dfrac{e^{3x}}{\sin^{-1} x - \cos x}$

$$f'(x) = \frac{[3e^{3x}](\sin^{-1}x - (\cos x)) - e^{3x}\left[\frac{1}{\sqrt{1-x^2}} + \sin x\right]}{(\sin^{-1}x - \cos x)^2}$$

(d) $f(x) = \ln\left(\sin^2(3x+2) + 1\right)$

$$f'(x) = \frac{1}{\sin^2(3x+2)+1}\left[2\sin(3x+2)\cos(3x+2)\,[3]\right]$$

(e) $f(x) = (\tan^{-1} x)^{\sqrt{x}}$

$$\ln y = \sqrt{x}\,\ln(\tan^{-1}x)$$

$$\cancel{y}\,\frac{1}{\cancel{y}}\,y' = y\left\{\frac{1}{2\sqrt{x}}\ln(\tan^{-1}x) + \sqrt{x}\left[\frac{1}{\tan^{-1}x}\left[\frac{1}{1+x^2}\right]\right]\right\}$$

$$\therefore f'(x) = (\tan^{-1}x)^{\sqrt{x}}\left\{\frac{\ln(\tan^{-1}x)}{2\sqrt{x}} + \frac{\sqrt{x}}{\tan^{-1}x\,(1+x^2)}\right\}$$

Value	
3	2. Find the equation of the tangent line to the curve $\sin x + \cos y = \sin x \cos y + (\sqrt{2} - \frac{1}{2})$ at the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

$$\cos x - \sin y \cdot y' = \cos x \cos y - \sin x \sin y \cdot y'$$

$$(\sin x \sin y - \sin y) y' = \cos x \cos y - \cos x$$

$$\therefore \quad y' = \frac{\cos x \cos y - \cos x}{\sin x \sin y - \sin y}$$

$$m = y'\Big|_{(\frac{\pi}{4}, \frac{\pi}{4})} = \frac{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}}$$

$$= 1$$

$\therefore$ tangent line $y = m(x - x_0) + y_0$

is

$$y = 1\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}$$

Value	
24	3. Find each of the following integrals:

(a) $\int \left(e^{2x} + 2^x + x^2 + \frac{2}{x} + 2\right) dx$

$= \frac{e^{2x}}{2} + \frac{2^x}{\ln 2} + \frac{x^3}{3} + 2\ln|x| + 2x + C$

(b) $\int_0^{\pi/4} \frac{\sec^2 x}{1+\tan x} \, dx$

$u = 1+\tan x$ $\qquad x = \frac{\pi}{4} \Rightarrow u = 1+\tan\frac{\pi}{4} = 2$

$du = \sec^2 x \, dx$ $\qquad x = 0 \Rightarrow u = 1$

$= \int_1^2 \frac{1}{u} \, du$

$= \ln|u| \Big|_1^2$

$= \ln 2 - \cancel{\ln 1}^{0}$

$= \ln 2$

(c) $\int \sin^{-1} x \, dx$

$u = \sin^{-1} x \qquad v = x$

$\int u \, dv = uv - \int v \, du \qquad du = \frac{1}{\sqrt{1-x^2}} dx \qquad dv = dx$

$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$ $\qquad w = 1-x^2$, $dw = -2x\,dx$, $-\frac{1}{2}dw = x\,dx$

$= x\sin^{-1} x + \frac{1}{2}\int w^{-1/2} dw$

$= x\sin^{-1} x + \frac{1}{\cancel{2}} \frac{\cancel{2}(1-x^2)^{3/2}}{3} + C$

Value

Find each of the following integrals:

(d) $\int \sin^2 x \cos^2 x \, dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\, dx$$

$$= \frac{1}{4}\int 1 - \left(\frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\int 1-\cos 4x \, dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + C$$

(e) $\int \frac{\sqrt{x^2-1}}{x}\, dx$

$x = \sec\theta$

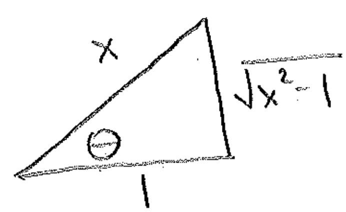
$dx = \sec\theta \tan\theta \, d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1}$

$= \tan\theta$

$$= \int \frac{\tan\theta}{\cancel{\sec\theta}} \cancel{\sec\theta} \tan\theta \, d\theta$$

$\sec\theta = \frac{x}{1}$



$$= \int \tan^2\theta \, d\theta$$

$$= \int \sec^2\theta - 1 \, d\theta$$

$$= \tan\theta - \theta + C$$

$$= \sqrt{x^2-1} - \sec^{-1} x + C$$

Value

Find the following integrals:

(f) $\int \frac{dx}{\sqrt{7+6x-x^2}}$

$-(x^2-6x+9)+7+9$

$16-(x-3)^2$

$= \int \frac{1}{\sqrt{16-(x-3)^2}} dx$

$= \sin^{-1}\left(\frac{x-3}{4}\right) + C$

(g) $\int \frac{2x^2-3x+1}{x^3+x}\,dx$

$\frac{2x^2-3x+1}{x(x^2+1)} = \frac{\overset{1}{A}}{x} + \frac{\overset{1}{B}x+\overset{-3}{C}}{x^2+1}$

$2x^2-3x+1 = A(x^2+1) + (Bx+C)x$

$x=0 \Rightarrow \boxed{1 = A}$

$2x^2-3x+1 = Ax^2+A+Bx^2+Cx$

$= (A+B)x^2 + Cx + A$

x^2: $A+B=2$ $A=1$ $\therefore \boxed{B=1}$

x: $\boxed{C=-3}$

$= \int \frac{1}{x}dx + \int \frac{x}{x^2+1}dx - 3\int \frac{1}{x^2+1}dx$

$u = x^2+1$

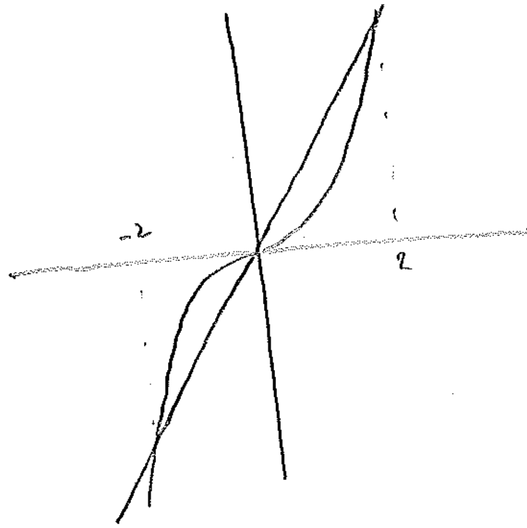
$\frac{1}{2}du = x\,dx$

$= \ln|x| + \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + C$

Value

3

4. Find the area of the region bounded by the curves $y = x^3$ and $y = 4x$.



intercepts

$x^3 = 4x$

$x^3 - 4x = 0$

$x(x+2)(x-2) = 0$

$\therefore x = -2, 0, 2$

$$\text{AREA} = \int_L^R [\text{TOP}(x) - \text{BOT}(x)]\,dx$$

$$= \int_{-2}^{0} x^3 - 4x\,dx + \int_0^2 4x - x^3\,dx$$

$$= 2\int_0^2 4x - x^3\,dx \quad \text{by symmetry}$$

$$= 2\left[\frac{4x^2}{2} - \frac{x^4}{4}\right|_0^2$$

$$= 2\left[\left(2(2)^2 - \frac{2^4}{4}\right) - 0\right]$$

$$= 2\left[8 - 4\right]$$

$$= 2(4)$$

$$= 8$$

Value	
8	5. Given that $f(x) = \dfrac{9(x-1)(x+3)}{(x-3)^2}$, $f'(x) = \dfrac{-72x}{(x-3)^3}$ and $f''(x) = \dfrac{72(2x+3)}{(x-3)^4}$,

(a) Calculate $\lim_{x\to 3^-} f(x)$, $\lim_{x\to 3^+} f(x)$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$.

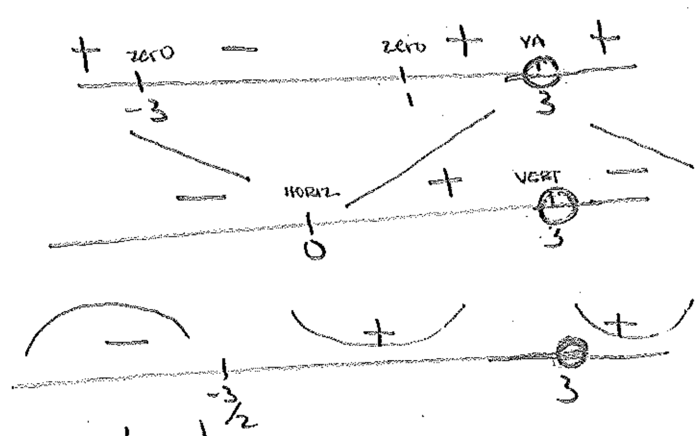
(b) Construct sign diagrams for $f'(x)$ and $f''(x)$. Identify the intervals where $f(x)$ is increasing/decreasing and concave up/concave down, using interval notation.

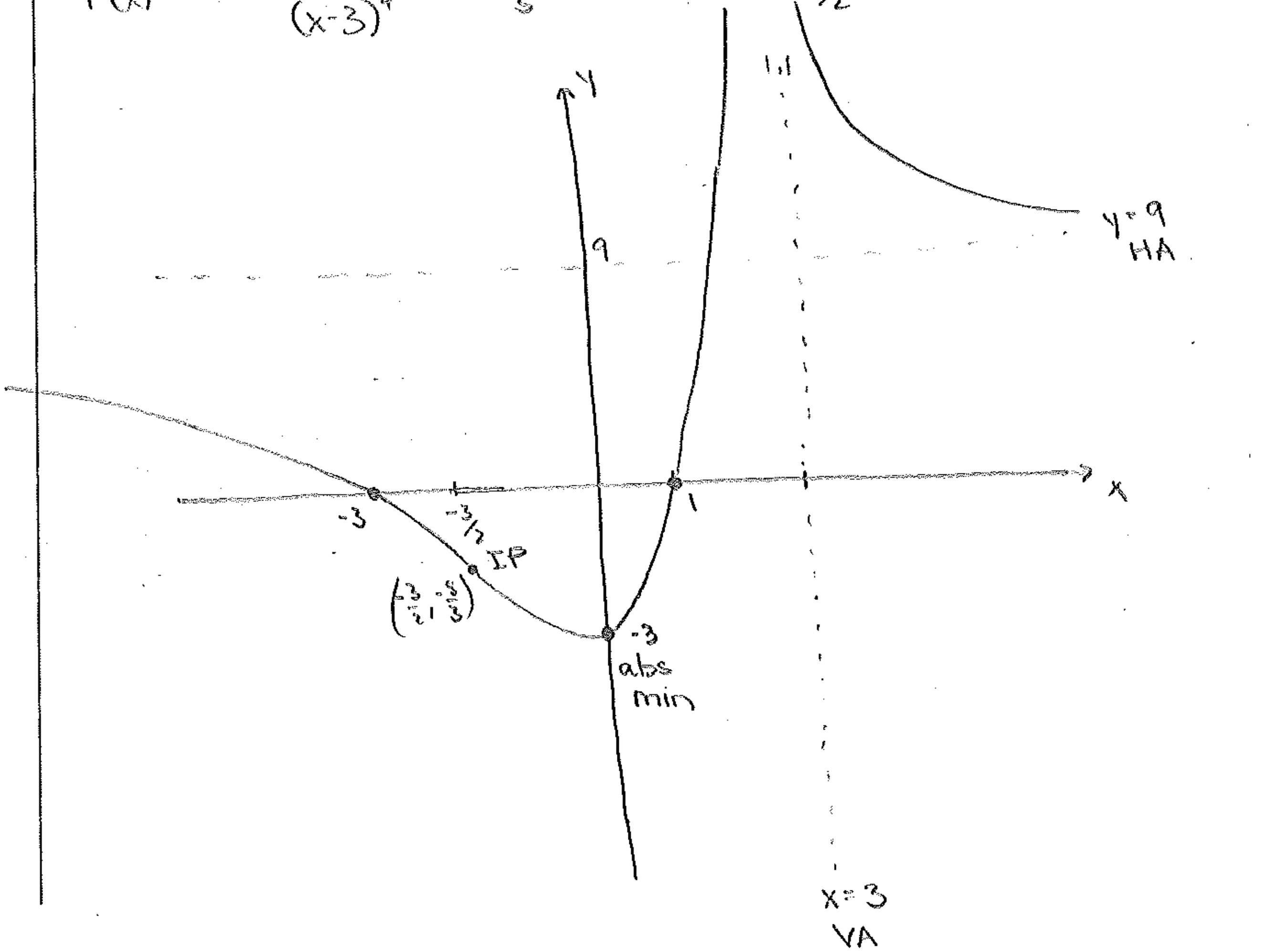
(c) Sketch the graph of f. Label all intercepts, local extreme values, all points of inflection, and any horizontal and vertical asymptotes.

$f(x) = \dfrac{9(x-1)(x+3)}{(x-3)^2}$ 1, -3 / 3

$f'(x) = \dfrac{-72x}{(x-3)^3}$ 0 / 3

$f''(x) = \dfrac{72(2x+3)}{(x-3)^4}$ $-3/2$ / 3





Value

(Use this page for Question 5)

$f(0) = \frac{\cancel{9}(-1)(3)}{\cancel{(-3)^2}} = -3 \quad (0,-3)$

$f(-\frac{3}{2}) = \frac{9(-\frac{5}{2})(\frac{3}{2})}{(-\frac{9}{2})^2} = \cancel{9}\left(\frac{-15}{\cancel{4}}\right)\left(\frac{\cancel{4}}{\cancel{9}\cdot 9}\right) = -\frac{5}{3}$

VA @ $x = 3$

$\lim_{x\to 3^-} f(x) = +\infty$
$\lim_{x\to 3^+} f(x) = +\infty$ } from sign diagram for $f(x)$

HA

$\lim_{x\to\pm\infty} \frac{9\frac{(x-1)}{x}\frac{(x+3)}{x}}{\frac{(x-3)^2}{x^2}}$

$= \lim_{x\to\pm\infty} \frac{9(1-\cancel{\frac{1}{x}}^{0})(1+\cancel{\frac{3}{x}}^{0})}{(1-\cancel{\frac{3}{x}}^{0})^2}$ $\quad \therefore y = 9$ is a H.A.

$= 9$

$f(x)$ is increasing on $(0,3)$
decreasing on $(-\infty,0)\cup(3,\infty)$
concave up on $(-\frac{3}{2},3)\cup(3,\infty)$
concave down on $(-\infty,-\frac{3}{2})$

Value	
4	6. (a) Let $f(x)$ be a function. State the definition of the derivative $f'(x)$.
	(b) Use the definition of the derivative to find the derivative of $f(x) = \dfrac{1}{3-2x}$.

a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided the limit exists

b) $$f'(x) = \lim_{h \to 0} \frac{\dfrac{1}{[3-2(x+h)]}\dfrac{(3-2x)}{(3-2x)} - \dfrac{1}{(3-2x)}\dfrac{[3-2x-2h]}{[3-2(x+h)]}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} - \cancel{2x} - \cancel{3} + \cancel{2x} + 2h}{(3-2(x+h))(3-2x)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{(3-2(x+h))(3-2x)} \cdot \frac{1}{\cancel{h}}$$

$$= \frac{2}{(3-2x)(3-2x)}$$

$$= \frac{2}{(3-2x)^2}$$

check by chain rule

$f(x) = (3-2x)^{-1}$

$f'(x) = -(3-2x)^{-2}[-2]$

$= \dfrac{+2}{(3-2x)^2}$ ✓

Value

5 7. 36π cm^3 of clay is being shaped on a pottery wheel. The clay is in the shape of a right circular cylinder. If the radius of the cylinder is decreasing at 1 cm per second, how fast is the height changing when the radius is 3 cm?

$h = \frac{36}{r^2} = \frac{36}{3^2} = 4$

$\frac{dr}{dt} = -1$ cm/s $\quad \frac{dh}{dt} = ?$

$V = \pi r^2 h = 36\pi$

$\therefore r^2 h = 36$

$2r\frac{dr}{dt} h + r^2 \frac{dh}{dt} = 0$

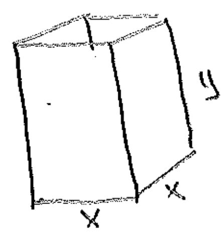
$\frac{dh}{dt} = -\frac{2\cancel{r} h \cdot \frac{dr}{dt}}{r^{\cancel{2}}}$

$\left.\frac{dh}{dt}\right|_{r=3} = \frac{-2(4)}{3}(-1)$

$= +\frac{8}{3}$ cm/s

The height is increasing at $\frac{8}{3}$ cm/s.

5 8. A box has a square base and no top. The total area of the sides and base of the box is 48 cm^2. Find the dimensions of the box so it has maximum volume.



$SA = x^2 + 4xy = 48$

$\therefore y = \frac{48 - x^2}{4x}$

$x, y > 0$

$x < \sqrt{48} = 4\sqrt{3}$

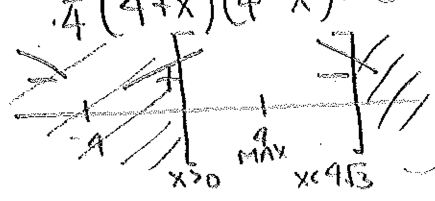
max $V = x^2 y$

$V = x^{\cancel{2}}\left(\frac{48 - x^2}{4\cancel{x}}\right)$

$V = \frac{1}{4}(48x - x^3)$

$V' = \frac{1}{4}(48 - 3x^2) = 0$

$\frac{3}{4}(4+x)(4-x) = 0$



$\therefore$ max volume when $x = 4$ cm & $y = \frac{48 - 4^2}{4(4)} = 2$ cm

ie the box should be

4cm × 4cm × 2cm

Value

9. Evaluate the limits. Do NOT use l'Hospital's Rule.

2 (a) $\lim_{x\to 1} \frac{(\sqrt{x^2+3}-2)}{(x^2-1)} \frac{(\sqrt{x^2+3}+2)}{(\sqrt{x^2+3}+2)}$ $\to \frac{0}{0}$ DNE $x-1$ will cancel

$= \lim_{x\to 1} \frac{x^2+3-4}{(x+1)(x-1)(\sqrt{x^2+3}+2)}$

$= \lim_{x\to 1} \frac{\cancel{x^2-1}\ 1}{\cancel{(x+1)(x-1)}(\sqrt{x^2+3}+2)}$

$= \frac{1}{2+2} = \boxed{\frac{1}{4}}$

2 (b) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\to \frac{0}{0}$

$= \lim_{x\to 0} \frac{(1-\cos x)}{x^2} \frac{(1+\cos x)}{(1+\cos x)}$

$= \lim_{x\to 0} \frac{(1-\cos^2 x)}{x^2} \cdot \frac{1}{(1+\cos x)}$

$= \lim_{x\to 0} \left[\left(\cancel{\frac{\sin x}{x}} \right)^{2} \left(\frac{1}{1+\cos x} \right) \right]$

$1^2 \cdot \frac{1}{1+1}$

$= \boxed{\frac{1}{2}}$

2 (c) $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ $\to 0 \cdot \sin(\infty)$

$-1 \le \sin\theta \le 1$ for all θ

so $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$

$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^2$

$0 \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \le 0$

$\therefore \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by squeeze theorem.

Value

10. Evaluate the limit. [You may use l'Hospital's Rule.]

2 (a) $\lim_{x\to 0} \frac{\sin^{-1}(2x)}{x}$ $\to \frac{0}{0}$

$\overset{H}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-(2x)^2}}(2)}{1}$

$= \frac{2}{\sqrt{1-0^2}} = \boxed{2}$

2 (b) $\lim_{x\to 0^+} x\ln(\sin x)$ $\to 0\ln(0^+) = 0\cdot(-\infty)$

$= \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} \to \frac{-\infty}{\infty}$

$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{-\frac{1}{x^2}}$

$= \lim_{x\to 0^+} \left(-\cancel{\frac{x}{\sin x}}^{1} \cdot x\cos x\right)$

$= -1\cdot 0\cdot 1 = \boxed{0}$

2 11. (a) State both parts of the Fundamental Theorem of Calculus.

I Given $f(x)$ is continuous on $[a,b]$.
Define $F(x) = \int_0^x f(t)\,dt$ for all $x \in [a,b]$

then $\frac{d}{dx}F(x) = f(x)$

II Given $f(x)$ is continuous on $[a,b]$
and $G(x)$ is any antiderivative of $f(x)$ on $[a,b]$
then $\int_a^b f(x)\,dx = G(b) - G(a)$

Value	
2	(b) Let $y = \int_0^{x^2} \sqrt{4t^3+2}\,dt$. Find $\frac{dy}{dx}$.

$$y' = \sqrt{4(x^2)^3+2}\,[2x]$$

3	12. Evaluate the integral $\int_{1/2}^{\infty} x^{-2}\,dx$.

$$= \lim_{t\to\infty} \int_{1/2}^{t} x^{-2}\,dx$$

$$= \lim_{t\to\infty} \frac{x^{-1}}{-1}\Big|_{1/2}^{t}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t} + 2\right]$$

$$= 0 + 2$$

$$= \boxed{2}$$

Value
6

13. Use the definition of the derivative to prove the following:

(a) The Product Rule. State the Product Rule before you prove it.

(b) $\dfrac{d}{dx}\cos x = -\sin x$.

a) Given $f(x)$ and $g(x)$ are differentiable then

$$[f(x)\cdot g(x)]' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Proof: $[f(x)\cdot g(x)]' = \lim\limits_{h\to 0} \dfrac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left\{ \left(\frac{f(x+h)-f(x)}{h}\right) g(x+h) + f(x)\left(\frac{g(x+h)-g(x)}{h}\right) \right\}$$

$$= \underbrace{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}_{f \text{ is diff.}} \ \underbrace{\lim_{h\to 0} g(x+h)}_{g \text{ is diff } \therefore \text{ cont.}} + \underbrace{\lim_{h\to 0} f(x)}_{\text{indep. of } h} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x}{h}}_{g \text{ is diff}}$$

$$= f'(x)\cdot g(x) + f(x)\, g'(x)$$

b) $\dfrac{d}{dx}\cos x = \lim\limits_{h\to 0} \dfrac{\cos(x+h) - \cos x}{h}$

$$= \lim_{h\to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h}$$

$$= \lim_{h\to 0} \left[\cos x \left(\cancelto{0}{\frac{\cosh - 1}{h}}\right) - \sin x \, \cancel{\frac{\sinh}{h}}\right]$$

$$= 0 - \sin x\,(1)$$

$$= -\sin x.$$

Value

3 14. Define what it means for f to be continuous at $x = a$. Prove that if f is differentiable at $x = a$, then f is continuous at $x = a$.

$f(x)$ is continuous at $x=a$ iff $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Proof of Theorem:

$$f(x) = f(x) - f(a) + f(a)$$

$$f(x) = \frac{f(x)-f(a)}{x-a}(x-a) + f(a) \quad \text{for } x \neq a$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a}(x-a) + \lim_{x\to a} f(a)$$

but f is diff @ $x=a$ · indep. of x

$$= f'(a)\underbrace{(a-a)}_{\to 0} + f(a)$$

$\therefore \lim_{x\to a} f(x) = f(a)$ Hence f is continuous @ $x=a$

3 15. Define the inverse sine function, $\sin^{-1} x$. Show that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

$y = \sin^{-1}x$ iff $x = \sin y$ for $x \in [-1,1]$ & $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Proof: let $y = \sin^{-1}x$

$\therefore$ by def'n $x = \sin y$ & differentiating implicitly

$$1 = \cos y \frac{dy}{dx}$$

OR $\frac{dy}{dx} = \frac{1}{\cos y}$ and $\cos^2 y + \sin^2 y = 1$

so $\cos y = \pm\sqrt{1-\sin^2 y}$

$= \frac{1}{\sqrt{1-\sin^2 y}}$ but $\cos y > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

so $\cos y = +\sqrt{1-\sin^2 y}$

$= \frac{1}{\sqrt{1-x^2}}$ $x = \sin y$

Value
3

16. (a) State the Mean Value Theorem for derivatives.

Given $f(x)$ is continuous on $[a,b]$ and differentiable of (a,b) then there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(b) Prove that if $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

let $x_1, x_2 \in (a,b)$ without loss of generality let $x_1 < x_2$

$f'(x) = 0$ for all $x \in (a,b)$ Hence f is differentiable

$\therefore$ continuous on the sub-interval $[x_1, x_2]$

$\therefore$ by the MVT there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

but $f'(c) = 0$ for all such c so

$$\cancel{(x_2-x_1)}\frac{f(x_2) - f(x_1)}{\cancel{(x_2 - x_1)}} = 0\,(x_2 - x_1) \qquad \text{since } x_2 - x_1 \neq 0$$

$$f(x_2) - f(x_1) = 0$$

or $f(x_2) = f(x_1)$

$\therefore f$ is constant on (a,b)

Total
100