Introduction to Calculus II
June 18, 2020

Rules for antiderivatives

- note: there is NO product rule or quotient rule or chain rule for antiderivatives
- constant multiple rule
  - given that k is an element of the reals, then the antiderivative of k of f(x) is going to be k times the antiderivative of f(x)
  - proof:
    - the definition of an antiderivative is that F'(x)=f(x) then F(x) is an antiderivative
      - professor notes these are short and easy proofs to write out but people often get lost in what we're doing because it looks like we're just writing the same things down over and over
    - let F(x) be an antiderivative of f(x) therefore if we take k times F(x) we end up with k times F(x)' which is kf(x)
    - so that means the antiderivative of k times f(x) has to be equal to k times F(x) +c (remember any time you take an antiderivative there has to be a constant)
    - but F(x) is our antiderivative, so we can say this is k times the antiderivative of f(x)
      - where is the c? Since any 2 antiderivatives differ by a constant, the notation inverse of x of f(x) can be considered to include the +c
      - you don't need to add the +c
- sum rule
  - antiderivative of f(x)+g(x) is antiderivative of f(x) plus the antiderivative of g(x)
  - proof:
    - let F(x) and G(x) be antiderivatives of f(x) and g(x) respectively
    - therefore if we take F(x)+G(x) and differentiate them by the sum rule, that's F(x)' + G(x)' which is just f(x)+g(x)
      - if the derivative of A is B, then the antiderivative of B must be A plus some c
    - so the antiderivative of f(x)+g(x) is F(x)+G(x)+c
    - we can then say by our definitions that this is the antiderivative of f(x) plus the antiderivative of g(x)
      - the c is swallowed up inside either of those – it disappears
- difference rule
  - antiderivative of f(x)-g(x) is the antiderivative of f(x) minus the antiderivative of g(x)
  - proof:
    - left for homework, the professor said
    - there are 2 ways to prove this
- caution: if I take the function f(x) times g(x) and we differentiate, we'll get f'(x) times g(x) plus f(x) times g'(x)
  - this means if we take the antiderivative of f'(x) times g(x) plus f(x) times g'(x) we get f(x) times g(x) +c
  - unfortunately, even if we have this particular layout (left side of the equation) we are unlikely to recognize it
  - hence the product rule for antiderivatives cannot be generalized
    - **there is no such rule that:** antiderivative of f(x) times g(x) = antiderivative of f(x) times antiderivative of g(x)

        - **same goes for the quotient rule and the chain rule – there is none for antiderivatives**
    - if we're given a product, quotient, or chain we will need to find some other way to evaluate the antiderivatives (this will be a big part of this course)
- professor repeated: there is no product rule, no quotient rule, and no chain rule for antiderivatives! Please remember this!

Evaluating antiderivatives

- find f(x):
    - 1. $f'(x) = x^2+1/x^2$
        - clean up the algebra first into a usable form
            - rewrite as $= x^2+x^{-2}$
            - $= x^3/3+x^{-1}/-1+c$ (or $x^3 – 1/x +c$)
    - 2. $f'(x) = x – {}^3\sqrt{x^4}+5$
        - rewrite as $x-x^{4/3}+5$
            - $= x^2-x^{7/3}+5x+c$
            - according to the rules you need something under the $x^{7/3}$: if I have the antiderivative of x^n is $x^{n+1}/n+1+c$, then I should divide by 7/3 – but who wants a fraction in the denominator?
            - So skip that step and say, it's going to be over 7 but I'll take the 3 that would be here and throw it straight up into the top: 3/7
            - S: does the 5x come from 5^0?
                - P: yes, if you have a constant and want to find its antiderivative, you have to realize the antiderivative of k = antiderivative of kx^0, so it becomes kx+c – this is what happened there
    - 3. find y:
        - $y'=1- 1/ \sqrt{x}$
        - rewrite: $= 1-x^{-1/2}$
        - $= x-2\ s^{1/2}/1+c$
    - 4. $y'=\sin x-1/\cos^2 x$ ← this is a quotient and there is no quotient rule!
        - Rewrite: $=\sin x-\sec^2 x$ (because 1/cos is sec)
        - recall: derivative of tan is $\sec^2$ so the antiderivative here should be what minus tanx+c?
            - you need to know the derivative of sinx is cosx but we know the derivative of cosx is -sinx – but here we're going backwards: trying to find antiderivatives not derivatives, so we have to work out which one will be negative
            - professor has taught this course for years and he still has no idea what the antiderivative of sinx is; he hasn't memorized it and doesn't intend to
                - we know sine will get us cosine, and cosine will get me sine; then work backwards, differentiating
                - if we take cos and differentiate it, we get -sin
                - but we're not supposed to have a minus here so just change this to be -cos
            - so: = -cosx-tanx+c
            - S: can you say why we can't be specific of what the antiderivative of sinx is?
                - P: there is an antiderivative – we'll come back to this later
            - we know:
                - sinx prime = cosx
                - antiderivative of cosx=-sinx+c

- cosx prime – sinx+c
- antiderivative of sinx=-cosx+c
- BUT remembering all these four and knowing which applies where, and where to put the negative signs, is a lot and gets confusing
  - so instead the professor doesn't care about the two antiderivatives above – as long as he knows derivative of cos and derivative of sin, that's all he needs to know
  - he knows a cos will get him a sin, and a sin will get him a cos: one automatically leads to the other – doesn't matter if you're differentiating or finding antiderivatives – BUT we don't know if it's supposed to be positive or negative
  - so, get the sign by differentiating
    - if you differentiate cosx, you end up with the sinx but negative
    - but in this question we don't want the negative sign, so we instead differentiate -cosx to get us a positive sinx

- 5. $y''=(x-2)^2$ ← remember no chain rule for antiderivatives!
  - Algebra is all we have for now, so expand:
    - notice we have y double prime so we have to find the antiderivative, and then find the antiderivative again
    - $=x^2-4x+4$
    - $=x^3/3-4x^2/2+4x+c$ (this is y')
    - do it again to get double prime
      - rewrite first: $y'=x^3/3-2x^2+4x+c$
        - note: we add a constant every time we find an antiderivative; two antiderivatives implies two constants
        - so here we can denote c_1
      - $y=x^4/3$ times $4 – 2x^3/3 + 4x^2/2 +c_1x+c$
        - c_1 is a constant and the antiderivative of $k=kx+c$ so antiderivative of $c_1=c_1+c_2$
        - the two constants don't need to be the same
      - rewrite: $y = x^4/12 – 2x^3/3 +2x^2 + c_1x + c_2$
        - S: can we use d as the second constant?
          - P: yes, but we tend to use c_1 and c_2 etc because you could run out of letters otherwise
      - professor noted that if you are able to skip a step or two that's okay, he's just writing everything out to make sure everybody understands

Initial value problems (IVP)

- these are also called differential equations (though these are the simplest differential equations you could come up with)
  - we offer two half courses in differential equations at University of Winnipeg – the equations you'd see there are much more difficult so don't assume these ones here are at all similar; the professor loved these courses and said they were very interesting
    - sample question: a snowplow leaves its shed at noon, travels 2 miles in the first hour and 1 mile in the second hour; at what time did it start snowing? You can figure this out with advanced differential equations!
- Type I (easier)
  - our constant c from the antiderivative is unknown

- it is replacing some constant (perhaps 0) lost in the differentiation process
  - for example, $y=x^2+5 \rightarrow y'=2x$; antiderivative: $y=2x^2/2+c = x^2+c$ (starting here, how would we know that's 5? there's no way to know)
- if we want to know the exact value of this lost c, we need extra information, called initial values

- example:
  - 1. $f'(x)=12x^2-4x+5$. Find f(x) if f(1)=8
    - the right side above is the initial value stuff
    - find the antiderivative:
      - $12x^3/3-4x^2/2+5x+c$
      - rewrite: $4x^3-2x+5x+c$ (this is f(x)
    - we know $f(1) = 4(1)^3-2(1)+5(1)+c=8$
      - rewrite: 4-2+5+c=8
      - =7+c=8 so c=1
    - so $f(x)=4x^3-2x+5x+1$
    - with the extra information we are able to figure out c=1 and plug it into the equation
  - 2. $f'(x)=2x+4/1+x^2$ given f(1)= ∏
    - realize that $4/1+x^2$ is just $\tan^{-1}x$ derivative
    - = $1^2+4\tan^{-1}(1)+c=$ ∏
      - think of the unit circle: tan of ∏/4=1
    - so we end up with c=-1
      - because 1 + ∏ + c = ∏ so c= -1
    - therefore $f(x)=2x^2+4\tan^{-1}x+1$
  - 3. $f'''(x)=6+3/2\sqrt{x^3}$ and we know f''(1)=-3 and f'(4)=4 and f(1)=-4
    - this is the 3rd derivative so we have to anti-differentiate three times (therefore 3 constants – so we need 3 initial values to get all the constants)
    - rewrite:
      - $f''' = 6+3/2x^{-3/2}$
      - $f'' = 6x+3x^{-1/2}+c_1 = 6x-3/\sqrt{x}+c_1$
      - $f''(1) = 6(1)-3/ \sqrt{1} + c_1 = -3$
        - so c_1=-3+3 or -6
    - rewrite:
      - $= 6x-3x^{-1/2}-6$
    - now find f':
      - $= 6x^2/2-3x^{1/2}/1-6x+c_2$
      - rewrite: $= 3x^2-6x^{1/2}-6x+c_2 \rightarrow 3x^2-6 x-6x+c_2$
      - $f'(4)=3(16)-6(2)-6(2)+c_2=4$
        - $= 48-12-24+c_2 = 4$
        - so $c_2=-8$
    - $f'(x)=3x^2-6x^{1/2}-6x-8$
    - therefore:
      - $3x^3/3-6x^{3/2}/3-6x^2/2-8x+c_3$
      - $= x^3-4x^{3/2}-3x^2-8x+c_3$
      - but we had f(1)=-1 so we'll get $f(1)=1^3-4(1)^{3/2}-3(1)^2-8(1)+c_3 = -4$
      - so $1-4-3-8+c_3=0$
      - so $c_3=10$

- therefore $f(x)=x^3-4x^{3/2}-3x^2-8x+10$
- note: in type I we are given initial values for f, f', f'', f''', ... so we can evaluate the c's as we go
- Type II
  - find f(x):
    - 1. $f''(x)=6x^2-12x$ and are given f(-1)=-5/2 and f(2)=-10
      - notice no f' value is given – this is why this is type II (here we need to calculate f(x) with 2 constants then solve the system
        - $f'(x)=2x^3-6x^2+c_1$
        - $f(x)=2x^4/4=6x^3/3+c_1x+c_2$
          - don't forget the x after the c_1 here!
        - $=x^4/2-2x^3+c_1x+c_2$
      - $f(-1)=1/2+2+c_1(-1)+c_2=-5/2$
        - so $c_1+c_2 = -5$ (let's call this equation A)
      - $f(2) = 8-16+2c_1+c_2=10$ (plug this into the f(x) formula above)
        - $2c_1+c_2=-2$ (let's call this equation B)
      - so add the negative of equation B to equation A
        - this is called elimination
        - elimination is almost always better than substitution
        - $3c_1=3$ therefore $c_1=1$
        - put it into equation A:
          - $-c_1+c_2=-5$ so $-1+c_2=5$ so $c_2=-4$
          - and $f(x) = x^4/2-2x^3+x-4$
    - 2. f''(x)=-2cosx+sinx for f(0)=4 and f(∏)=3∏
      - I know derivative of cos=sin; once I figure out if it's positive or negative, I know the derivative sine is cosine
      - $f'(x)=-2sinx-cosx+c_1$
      - $f(x)=2cosx-sinx+c_1+c_2$
      - rewrite: $2cosx-sinx+c_1x+c_2$
      - = c=4-2=2
      - so f(x)=2cosx-sinx
      - S: how did -2sinx become 2cosx?
        - P: go back to the start of the question – it's a two-step process. I know cos will get me sin and sin will get me cos. The question is, positive or negative. I take cosx and differentiate it to go backwards: so the derivative of sine is cosine so that's fine. But for the next one if I differentiate that, the derivative of cosx is -sinx but we don't have our minus so we have to introduce the minus to the -cosx
      - f(∏)=2cos ∏ -sin ∏ +c_1 ∏ +2 = 3 ∏
        - cos of pi is -1, sine of pi is 0 so we end up with -2 + ∏ times c_1+2 = 3 ∏
        - so $c_1=3$ because the 2's cancel
        - therefore f(x)=2cosx-sinx+3x+2
    - note: given $y^{(4)}=x^2$ find y
      - $y^{(3)}=x^3/3+c_1$
      - $y''=x^4/12+c_1x+c_2$
      - $y'=x^5/60+c_1x^2/2+c_2x+c_3$
      - $y=x^6/360+c_1x^3/6+c_2x^2/2+c_3x+c_4$

        - notice the above is a polynomial of degree 3
            - 4 antiderivatives should give 4 terms in polynomial degree 3
            - 2 antiderivatives → 2 terms, polynomial degree 1
            - 12 antiderivatives → 12 terms, polynomial degree 11
    - too often, students get $x^6/360+c_1+c_2$... but that's just one constant – they're all the same, if you just take some unknown number plus another unknown number
    - whereas as above you can't combine the pieces because they all have different exponents of x – and that comes about by finding each and every antiderivative

Practice questions

- from the textbook, section 4.9
    - questions 1-12, 15, 17, 18, 20-22 (basic antiderivatives)
    - questions 25, 26, 29-37, 39-41, 43, 46, 48 (IVP)
    - questions 49, 51, 52 (miscellaneous)
    - questions 59-69 (everybody should read them but if you're physics students you should try them)

note: professor will post a sheet on solving systems and equations

- there are 4 examples; all are taken from from the professor's previous Intro to Calculus II final exams
- these are the kinds of questions you'll be expected to be able to answer on an exam

reminder: office hours later this morning; send the professor an email in advance to let him know you have questions

questions

- s: can we prove the difference rule by multiplying by -1 and then use the sum rule?
    - P: there are 2 ways, as I said – one way is to do what we did in the proof but change every sign to the minus sign; the other way is what you're suggesting so yes you can do that
    - I prefer using this way because the way we did it in class through the proof is ugly
- S2: can we use another way to solve systems equations for example using matrices?
    - P: yes, but the kinds of questions we get were developed through partial fraction expansions – so you'll find these things come in particular patterns. They aren't just randomly generated. When we try to do systems of equations there's a process we follow that's very step by step – and if you have no pattern that's the best way to go. But if the equations have a relationship there is a nicer way to get there by using elimination – which is in a way what matrices is, but you're going through it in a formal way. If you try doing these by straight elimination it's faster and easier than matrices, but matrices are way easier than doing it by elimination
    - I recommend you try it by matrices, and by elimination, and by substitution – but the latter is very hard and it's awful. I think I've only ever had one or two students get it right by substitution however